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TITLE

ON A NEW FUNCTIONAL TRANSFORM IN ANALYSIS: THE MAXIMUM TRANSFORM

by

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ON A NEW FUNCTIONAL TRANSFORM IN ANALYSIS: THE MAXIMUM TRANSFORM

SUMMARY

In the study of mathematical economics and operations research, we encounter the problem of determining the maximum of the function

$$F(x_1, x_2, \ldots, x_N) = f_1(x_1) + f_2(x_2) + \ldots + f_N(x_N)$$

over a region R defined by $x_1 + x_2 + \ldots + x_N = x$, $x_i \geq 0$. Under various assumptions concerning the f_i, this problem can be studied analytically, and it can also be treated analytically by means of the theory of dynamic programming.

It is natural in this connection to introduce a "convolution" of two functions f and g, $h = f * g$, defined by

$$h(x) = \max_{0 \leq y \leq x} [f(y) + g(x - y)].$$

For purposes of general study, it is more convenient to introduce instead the convolution $h = f \otimes g$ defined by

$$h(x) = \max_{0 \leq y \leq x} [f(y)g(x - y)].$$

It is easy to see that the operation $\otimes$ is commutative and associative provided that all functions involved are nonnegative. By analogy with the relation between the Laplace transform and the usual convolution,

$\int_0^x f(y)g(x - y)dy$, it is natural to seek a functional transform

$$M(f) = F$$

with the property that

$$M(f \otimes g) = M(f)M(g),$$

that is,

$$H(z) = F(z)G(z)$$

where H, F, G are the transforms of h, f, g respectively.

We shall show that M exists and has a very simple form. In addition, M^{-1} has a very simple and elegant representation in a number of cases. More detailed discussions and extensions will be presented subsequently.

1. Introduction

In the study of mathematical economics and operations research, we encounter the problem of determining the maximum of the function

$$(1) \qquad F(x_1, x_2, \ldots, x_N) = f_1(x_1) + f_2(x_2) + \ldots + f_N(x_N)$$

over the region R defined by $x_1 + x_2 + \ldots + x_N = x$, $x_i \geq 0$. Under various assumptions concerning the f_i, this problem can be studied analytically; cf. Karush [1], [2], and it can also be treated analytically by means of the theory of dynamic programming [3].

It is natural in this connection to introduce a "convolution" of two functions f and g, $h = f * g$, defined by

$$(2) \qquad h(x) = \max_{0 \leq y \leq x} [f(y) + g(x - y)].$$

For purposes of general study, it is more convenient to introduce instead the convolution $h = f \otimes g$ defined by

$$(3) \qquad h(x) = \max_{0 \leq y \leq x} [f(y)g(x - y)].$$

It is easy to see that the operation $\otimes$ is commutative and associative provided that all functions involved are nonnegative. By analogy with the relation between the Laplace transform and the usual convolution, $\int_0^x f(y)g(x - y)dy$, it is natural to seek a functional transform

$$(4) \qquad M(f) = F$$

with the property that

$$(5) \qquad M(f \otimes g) = M(f)M(g),$$

that is,

$$(6) \qquad H(z) = F(z)G(z)$$

where H, F, G are the transforms of h, f, g respectively.

We shall show that M exists and has a very simple form. In addition,

M^{-1} has a very simple and elegant representation in a number of cases. More detailed discussions and extensions will be presented subsequently.

2. The Maximum Transform

Let a transform (1.4) be defined by the equation

$$(1) \qquad F(z) = \max_{x \geq 0} [e^{-xz} f(x)], \quad z \geq 0.$$

It will be assumed that $f(x)$ is continuous and nonnegative for $x \geq 0$. Furthermore, since $F(z)$ is unchanged when f is replaced by its monotone envelope, we shall consider (1) only for monotone nondecreasing f.

It is now a straightforward matter to prove (1.5) by the method used in the usual convolution. We have

$$(2) \qquad \begin{aligned} H(z) &= \max_{x \geq 0} \left[e^{-xz} \max_{0 \leq y \leq x} [f(y)g(x-y)] \right] \\ &= \max_{x \geq 0} \max_{0 \leq y \leq x} [e^{-xz} f(y) g(x-y)] = \max_{y \geq 0} \max_{x \geq y} [\quad] \\ &= \max_{y \geq 0} \left[f(y) \max_{x \geq y} [e^{-xz} g(x-y)] \right] \\ &= \max_{y \geq 0} \left[e^{-yz} f(y) \max_{w \geq 0} [e^{-wz} g(w)] \right] \\ &= \max_{y \geq 0} [e^{-yz} f(y)] \cdot \max_{w \geq 0} [e^{-wz} g(w)] = F(z)G(z) \end{aligned}$$

as desired.

To ensure the existence of $F = M(f)$ for $z > 0$, it is sufficient to assume that f satisfies a relation of the form $f(x) = O[x^c]$ for $x \geq 0$ where $c \geq 0$. The transform F is decreasing and continuous for $z > 0$; if $c = 0$, this holds for $z \geq 0$.

3. Inverse Operator

The determination of the existence and uniqueness of M^{-1} is of some complexity, and at this time we shall consider only special cases. If for $z > 0$, the maximum of $f(x)e^{-xz}$ can be found by differentiation, we have the maximizing value the equation $f'(x) - zf(x) = 0$. Suppose that this equation possesses a unique solution $x = x(z)$ with $dx/dz \neq 0$ (and hence < 0). For this value of x, we have $F(z) = e^{-xz}f(x)$. Differentiating this relation with respect to x, we have

$$(1) \qquad F'(z)\frac{dz}{dx} = \left(f'(x) - zf(x)\right)e^{-xz} - xf(x)e^{-xz}\frac{dz}{dx}$$

$$= - xf(x)e^{-xz}\frac{dz}{dx}.$$

Hence,

$$(2) \qquad x = - F'(z)/F(z), \qquad \text{or} \qquad F'(z) + xF(z) = 0.$$

But this is precisely the relation which gives the z minimizing $F(z)e^{xz}$, for fixed x. Hence, we have

$$(3) \qquad f(x) = \min_{z \geq 0} e^{xz}F(z),$$

the required inversion relation.

A simpler way to obtain this relation is the following. By (2.1), we have, for $x \geq 0$,

(4) $$F(z) \geq e^{-xz} f(x),$$

whence $F(z)e^{xz} \geq f(x)$. If there is a one-to-one correspondence between x and z values, we have $\min_{z \geq 0} F(x)e^{zx} \geq f(x)$, with equality for one value, whence (3).

4. Application

Let

(1) $$f(x) = \max_{R} [f_1(x_1)f_2(x_2)\dots f_N(x_N)],$$

where R is as in (1.1). Then, inductively,

(2) $$M(f) = \prod_{i=1}^{N} M(f_i), \qquad \text{or} \qquad F(z) = \prod_{i=1}^{N} F_i(z),$$

whence formally

(3) $$f(x) = \min_{z \geq 0} \left[e^{xz} \prod_{i=1}^{N} F_i(z) \right].$$

Similarly, if we have a "renewal" equation

(4) $$f(x) = a(x) + \max_{0 \leq y \leq x} [f(y)g(x - y)],$$

we have a formal solution

(5) $$f(x) = \min_{z \geq 0} \left[\frac{e^{xz} A(z)}{1 - G(z)} \right],$$

where $A = M(a)$, $G = M(g)$.

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1. W. Karush, "A queuing model for an inventory problem," Operations Research, Vol. 5, 1957, pp. 693-703

2. ________, "A general algorithm for the optimal distribution of effort," TM-616, System Development Corporation, May 1961

3. R. Bellman, "Dynamic programming," Princeton University Press, Princeton, New Jersey, 1957.